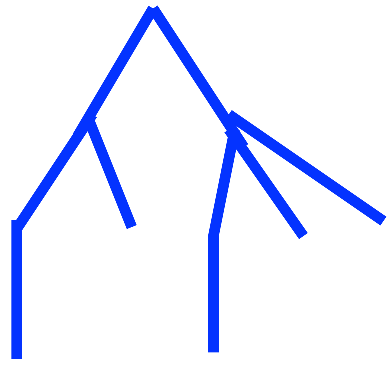
# Advanced Data Structures

Succinct Data Structures

# Arbitrary Ordered Trees

- Use parenthesis notation
- Represent the tree



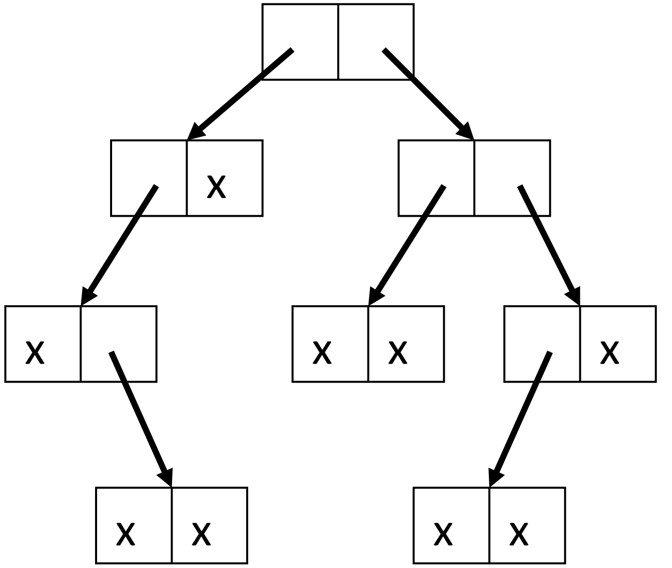
- As the binary string ((())())((())()()): traverse tree as "( " for node, then subtrees, then ")"
- 2 Bits per node

# Space for trees

- The space used by the tree structure could be the dominating factor in some applications.
  - Eg. More than half of the space used by a standard suffix tree representation is used to store the tree structure.
- Standard representations of trees support very few operations. To support other useful queries, they require a large amount of extra space.

# Standard representation

Binary tree: each node has two pointers to its left and right children




An n-node tree takes
2n pointers or 2n lg n bits

Supports finding left child or right child of a node (in constant time).

For each extra operation (eg. parent, subtree size) we have to pay, roughly, an additional n lg n bits.

# Can we improve the space bound?

- There are less than $2^{2n}$ distinct binary trees on $n$ nodes.
- $2n$ bits are enough to distinguish between any two different binary trees.
- Can we represent an $n$ node binary tree using $2n$ bits?

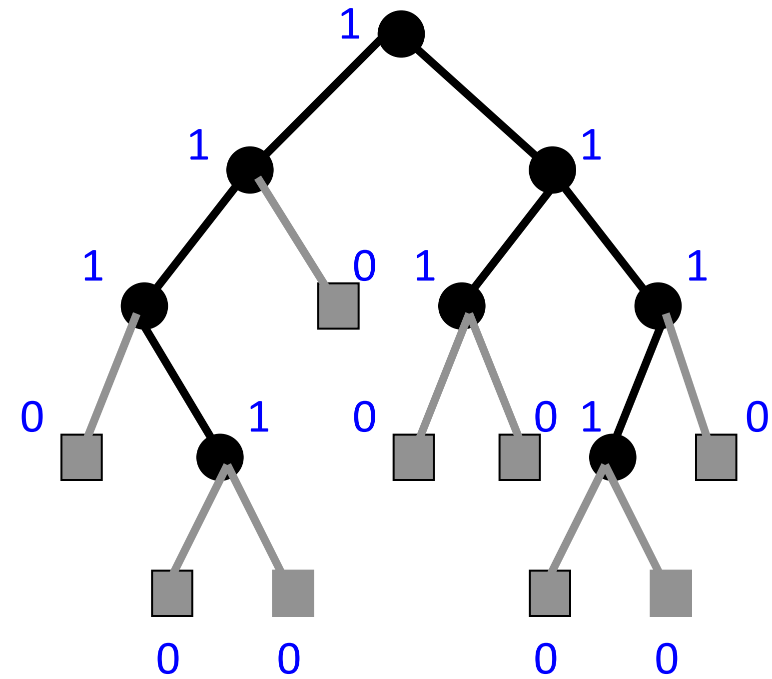
# Heap-like notation for a binary tree

Add external nodes

Label internal nodes with a 1
and external nodes with a 0

Write the labels in level order

1 1 1 1 0 1 1 0 1 0 0 1 0 0 0 0 0



One can reconstruct the tree from this sequence

An $n$ node binary tree can be represented in $2n+1$ bits.

What about the operations?

# Heap-like notation for a binary tree

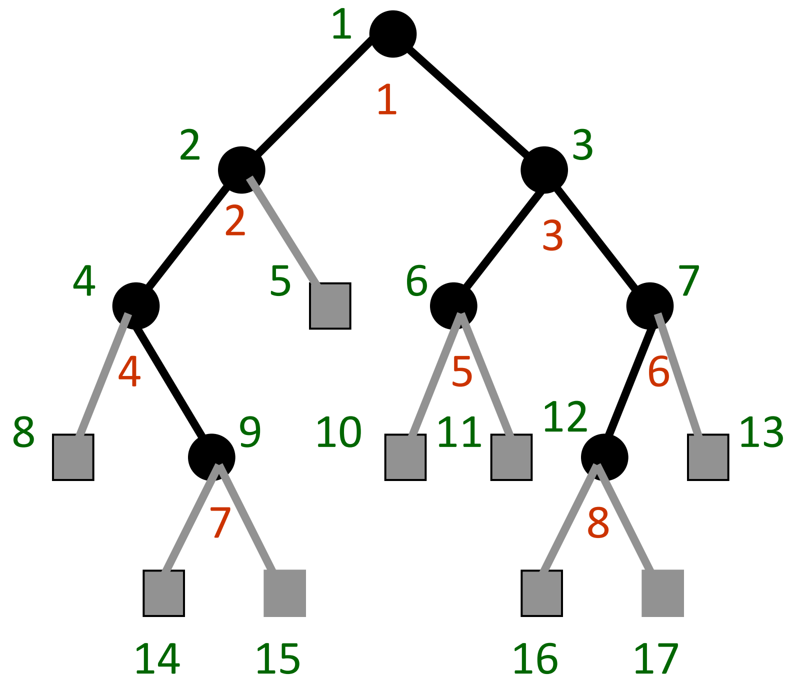
left child(x) = [2x]

right child(x) = [2x+1]

parent(x) = [⌊x/2⌋]

x → x: # 1's up to x

x → x: position of x-th 1




1 2 3 4 5 6 7 8

1 1 1 1 0 1 1 0 1 0 0 1 0 0 0 0 0

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

# Rank/Select on a bit vector

Given a bit vector B

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| B: | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |

$rank_1(i)$ = # 1's up to position i in B

$select_1(i)$ = position of the i-th 1 in B

(similarly $rank_0$ and $select_0$)

$rank_1(5) = 3$
$select_1(4) = 9$
$rank_0(5) = 2$
$select_0(4) = 7$

> Given a bit vector of length n, by storing an additional o(n)-bit structure, we can support all four operations in constant time.

An important substructure in most succinct data structures.

Have been implemented.

# Binary tree representation

- A binary tree on $n$ nodes can be represented using $2n+o(n)$ bits to support:
  - parent
  - left child
  - right child

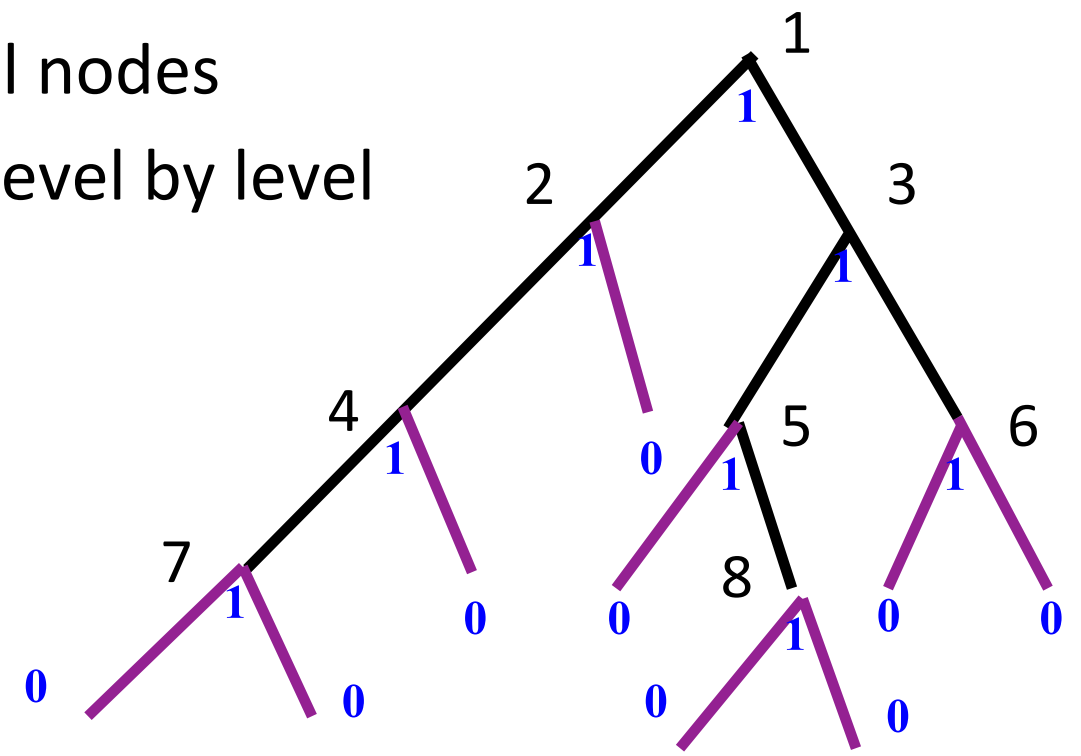
  in constant time.

- 1 1 1 1 0 1 1 1 0 0 1 0 0 0 0 0 0

# Heap-like Notation for a Binary Tree

Add external nodes

Enumerate level by level




1 2 3 4 5 6 7 8

Store vector 1 1 1 1 0 1 1 1 0 0 1 0 00000 length 2n+1

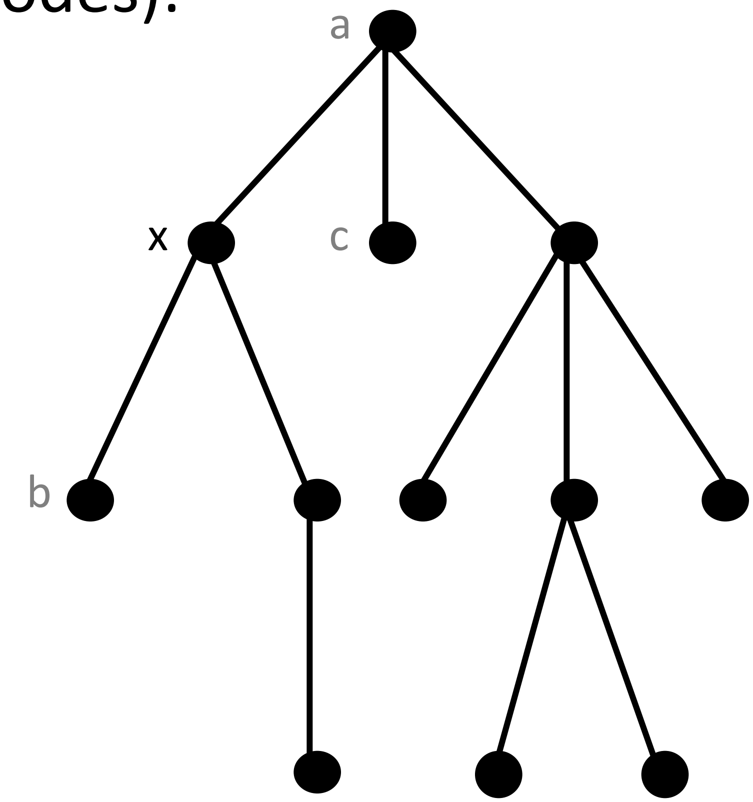
1 2 3 4 5 6 7 8 9 0 1 2 34567

# Ordered trees

A rooted ordered tree (on n nodes):

Navigational operations:
- parent(x) = a
- first child(x) = b
- next sibling(x) = c

Other useful operations:
- degree(x) = 2
- subtree size(x) = 4



# Ordered trees

- A binary tree representation taking 2n+o(n) bits that supports parent, left child and right child operations in constant time.
- There is a one-to-one correspondence between binary trees and rooted ordered trees
- Gives an ordered tree representation taking 2n+o(n) bits that supports first child, next sibling (but not parent) operations in constant time.
- We will now consider ordered tree representations that support more operations.

# Level-order degree sequence

Write the degree sequence in level order

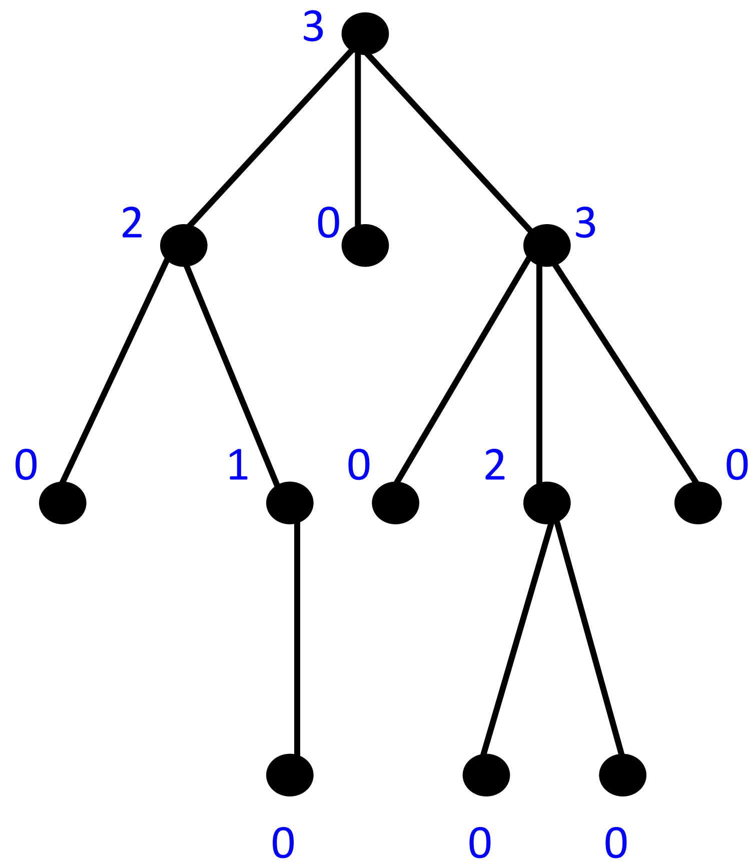
3 2 0 3 0 1 0 2 0 0 0 0

But, this still requires n lg n bits

Solution: write them in unary

1 1 1 0 1 1 0 0 1 1 1 0 0 1 0 0 1 1 0 0 0 0 0

Takes 2n-1 bits




A tree is uniquely determined by its degree sequence

# Supporting operations

Add a dummy root so that each node has a corresponding 1

1 0 1 1 1 0 1 1 0 0 1 1 1 0 0 1 0 0 1 1 0 0 0 0 0

1 2 3 4 5 6 7 8 9 10 11 12

node k corresponds to the
k-th 1 in the bit sequence

parent(k) = # 0's up to the k-th 1

children of k are stored after the k-th 0

supports: parent, i-th child, degree

(using rank and select)